An equivalence between the convergences of Ishikawa, Mann and Picard iterations

ŞTEFAN M. ŞOLTUZ*

Abstract. *We will show that the convergence of Picard iteration is equivalent to the convergence of Mann and Ishikawa iterations, when the operator is a contraction and aymptotic nonexpansive.*

Key words: *Picard iteration, Ishikawa iteration, Mann iteration, Ishikawa type iteration, Mann type iteration, contractive map, asymptotic nonexpansive map*

AMS subject classifications: 47H10

Received December 17, 2002 Accepted January 12, 2003

1. Introduction

Let X be a normed space. Let B be a nonempty, convex subset of X. Let $T : B \to B$ be a contraction with constant $L \in (0, 1)$. Let $x_1, u_1, v_1 \in B$ be three arbitrary points. We consider the following iteration, see [4]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ n = 1, 2, \dots . \tag{1}$$

The sequence $(\alpha_n)_n$ from $(0, 1)$ is convergent such that $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Iteration (1) is known as Mann iteration. Also, we consider the *Picard iteration*

$$u_{n+1} = T u_n, \ n = 1, 2, \dots . \tag{2}$$

The following iteration is known as *Ishikawa iteration*:

$$\begin{aligned} v_{n+1} &= (1 - \alpha_n)v_n + \alpha_n T w_n, \\ w_n &= (1 - \beta_n)v_n + \beta_n T v_n, \ n = 1, 2, \dots . \end{aligned} \tag{3}$$

The sequences $(\alpha_n)_n, (\beta_n)_n$ from $(0, 1)$, verify $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Ishikawa iteration is introduced in [2]. For $\beta_n = 0, \forall n \in \mathbb{N}$, Ishikawa iteration becomes Mann iteration.

The aim of this note is to prove an equivalence between the convergence of the above three iterations, when T is a contraction. When T is not a contraction, these

* "T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68-1, 3400 Cluj-Napoca, Romania, e-mail: `soltuzul@yahoo.com`, `soltuz@itwm.fhg.de`, `stefansoltuz@personal.ro`

iterations may have different behaviors. For instance, there exists an example, see [5], in which Mann iteration is not convergent, while Ishikawa iteration converges.

Let us consider the *Mann type iteration:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n. \tag{4}$$

The sequence $(\alpha_n)_n \subset (0,1)$, is convergent, $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. We consider the *Ishikawa type iteration*:

$$\begin{aligned} v_{n+1} &= (1-\alpha_n)v_n + \alpha_n T^n w_n, \\ w_n &= (1-\beta_n)v_n + \beta_n T^n v_n, \quad n = 1,2,\dots . \end{aligned} \tag{5}$$

The sequences $(\alpha_n)_n, (\beta_n)_n \subset (0,1)$, are convergent such that $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, and $\sum_{n=1}^{\infty}\alpha_n = \infty$.

The map T is said to be *asymptotically nonexpansive* if there exists a nonnegative sequence $(k_n)_n$, we take here $k_n \in (0,1), \forall n \in \mathbb{N}$, with

$$\lim_{n\to\infty} k_n = 1,$$

such that

$$\|T^n x - T^n y\| \le k_n \|x-y\|, \qquad \forall x,y \in B, \quad \forall n \in \mathbb{N}. \tag{6}$$

We prove also the equivalence between the convergence of iterations (4) and (5) for this kind of asymptotic nonexpansive operators.

The following lemma can be found in [9] as Lemma 4. Also, it can be found in [10] as Lemma 1.2, with another proof. A more general case is in Lemma 2 from [3]. In [1] it can be found as Lemma 2.

Lemma 1 [[1], [9], [10]]. *Let $(\rho_n)_n$ be a nonnegative real sequence satisfying*

$$\rho_{n+1} \le (1-\lambda_n)\rho_n + \sigma_n, \tag{7}$$

where $(\lambda_n)_n \subset (0,1)$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\sigma_n > 0, \forall n \ge 1$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty}\rho_n = 0$.

The following lemma is from [8].

Lemma 2 [[8]]. *Let $(\beta_n)_n$ be a nonnegative sequence such that $\beta_n \in (0,1]$, $\forall n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}\beta_n = \infty$, then $\prod_{n=1}^{\infty}(1-\beta_n) = 0$.*

2. The case when the map is a contraction

We are able now to give the following result:

Theorem 1. *Let X be a normed space, and B a nonempty convex subset of X. Let $T : B \to B$ be a contraction with constant $L \in (0,1)$. Suppose that there exists $x^* \in B$ such that $Tx^* = x^*$, and let $u_1 = x_1 \in B$. If the Picard iteration $(u_n)_n$ given by (2) strongly converges to x^* and $\|u_{n+1} - u_n\| = o(\alpha_n)$; then the Mann sequence $(x_n)_n$ given by (1) strongly converges to x^*. Conversely, if the Mann sequence $(x_n)_n$ given by (1) strongly converges to x^*, then the Picard iteration $(u_n)_n$ given by (2) strongly converges to x^*.*

Proof. From (1) and (2), we have $u_{n+1} = Tu_n,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$; thus, we get

$$x_{n+1} - u_{n+1} = (1-\alpha_n)(x_n - Tu_n) + \alpha_n(Tx_n - Tu_n).$$

Hence, we obtain

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le (1-\alpha_n)\|x_n - Tu_n\| + \alpha_n \|Tx_n - Tu_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + (1-\alpha_n)\|u_n - Tu_n\| + \alpha_n L \|x_n - u_n\| \\
&\le (1-\alpha_n(1-L))\|x_n - u_n\| + (1-\alpha_n)\|u_n - Tu_n\| \\
&= (1-\alpha_n(1-L))\|x_n - u_n\| + (1-\alpha_n)\|u_{n+1} - u_n\|.
\end{aligned}$$

We denote by $\rho_n := \|x_n - u_n\|,\ \lambda_n := \alpha_n(1-L) \in (0,1), \sigma_n := (1-\alpha_n)\|u_{n+1} - u_n\|,$ for all $n \in \mathbb{N}$, and we get (7). The assumptions of *Lemma 1* are fulfilled, and consequently we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

But $\lim_{n\to\infty} u_n = x^*$, for an $\varepsilon > 0$ there exists n_0 sufficiently large such that for $\forall n \ge n_0$, we have

$$\|x_n - u_n\| < \frac{\varepsilon}{2},\ \|u_n - x^*\| < \frac{\varepsilon}{2}.$$

Thus $\lim_{n\to\infty} x_n = x^*$, because

$$\|x_n - x^*\| \le \|x_n - u_n\| + \|u_n - x^*\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \ge n_0.$$

Conversely, we suppose that Mann iteration converges to x^*, and we prove that Picard iteration converges to x^*. The following implication is true

$$\lim_{n\to\infty} x_n = x^* \Rightarrow \lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{8}$$

We prove the implication (8). We can see that

$$\begin{aligned}
\|u_{n+1} - x_{n+1}\| &= \|(1-\alpha_n)(x_n - Tu_n) + \alpha_n(Tx_n - Tu_n)\| \\
&\le (1-\alpha_n)\|x_n - Tu_n\| + \alpha_n\|Tx_n - Tu_n\| \\
&\le \alpha_n L\|x_n - u_n\| + (1-\alpha_n)\left[\|x_n - x^*\| + \|x^* - Tu_n\|\right] \\
&\le \alpha_n L\|x_n - u_n\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)L^n\ \|x^* - u_1\|.
\end{aligned}$$

We denote by

$$a_n := \|x_n - u_n\|, \quad \beta_n := (1-\alpha_n)\left[\|x_n - x^*\| + L^n\ \|x^* - u_1\|\right], \quad \forall n \in \mathbb{N}.$$

Thus, we have $(a_n)_n$ a nonnegative sequence which verifies

$$a_{n+1} \le \alpha_n L a_n + \beta_n, \quad \forall n \in \mathbb{N}.$$

We note that $L \in (0,1) \Rightarrow \lim_{n\to\infty} L^n = 0$; also, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$, and consequently $\lim_{n\to\infty} \beta_n = 0$. It is easy to see that

$$\lim_{n\to\infty} a_n = 0, \text{ i.e. } \lim_{n\to\infty} \|x_n - u_n\| = 0.$$

We consider now the proof of the converse. For an $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\|u_n - x_n\| < \frac{\varepsilon}{2}, \ \|x_n - x^*\| < \frac{\varepsilon}{2}.$$

Finally, we get

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \|u_n - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ i.e. } \lim_{n\to\infty} u_n = x^*.$$

□

The convergence of $(u_n)_n$ is not a consequence of the Picard-Banach theorem. There the set B is closed. Here B is just a nonempty and convex set.

When T is a contraction, the Mann iteration is convergent if and only if the Ishikawa iteration is convergent. This is Theorem 3 from [7].

Theorem 2 [[7]]. *Let X be a normed space, and B a nonempty convex subset of X. Let $T : B \to B$ be a contraction with constant $L \in (0,1)$. Suppose that there exists $x^* \in B$ such that $Tx^* = x^*$. Let $x_1 = v_1 \in B$. The following two assertions are equivalent:*

(i) The Mann iteration $(x_n)_n$ given by (1) strongly converges to x^*,

(ii) The Ishikawa iteration $(v_n)_n$ given by (3) strongly converges to x^*.

Theorem 1 and *Theorem 2* lead us to the following result:

Corollary 1. *Let X be a normed space, and B a nonempty convex subset of X. Let $T : B \to B$ be a contraction with constant $L \in (0,1)$. Suppose that there exists $x^* \in B$ such that $Tx^* = x^*$. Let $p_1 = x_1 = v_1 \in B$. If the Picard iteration $(u_n)_n$ given by (2) strongly converges to x^*, and $\|u_{n+1} - u_n\| = o(\alpha_n)$, then the Mann sequence $(x_n)_n$ given by (1) strongly converges to x^* and the Ishikawa iteration $(v_n)_n$ given by (3) also strongly converges to x^*. Conversely, if the Mann sequence $(x_n)_n$ given by (1) strongly converges to x^* or the Ishikawa iteration $(v_n)_n$ given by (3) strongly converges to x^*, then the Picard iteration $(u_n)_n$ given by (2) strongly converges to x^*.*

There exists a case in which the assumption $\|u_{n+1} - u_n\| = o(\alpha_n)$ is fulfilled as we can see from the following remark:

Remark 1. *When $\alpha_n = 1/n, \forall n \geq 1$, then we have $\|u_{n+1} - u_n\| = o(1/n)$.*

Proof. We know $\|u_{n+1} - u_n\| \leq L^{n-1} \|u_2 - u_1\|$. Because $\lim_{n\to\infty} L^{n-1} n = 0$, we conclude that $\|u_{n+1} - u_n\| = o(1/n)$. □

3. The asymptotic nonexpansive case

For iteration (4), we are able now to give the following result:

Theorem 3. *Let X be a normed space, and B a nonempty convex subset of X. Let $T : B \to B$ be an asymptotic nonexpansive operator with $k_n \in (0,1)$. Suppose that there exists $x^* \in B$ such that $Tx^* = x^*$, and let $u_1 = x_1 \in B$. If the Picard iteration $(u_n)_n$ strongly converges to x^*, and $\|u_{n+1} - u_n\| = o(\alpha_n (1 - k_n))$, where $(\alpha_n)_n$ is the sequence from (2); then the Mann type sequence $(x_n)_n$ from (4),* *strongly converges to* x^*. *Conversely, if the Mann type sequence* $(x_n)_n$ *from (4) strongly converges to* x^*, *then the Picard iteration* $(u_n)_n$ *strongly converges to* x^*.

Proof. Suppose that Picard iteration converges, we will prove that Mann iteration converges. From (2) and (4), we have $u_{n+1} = Tu_n$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$; thus, we get

$$x_{n+1} - u_{n+1} = (1-\alpha_n)(x_n - Tu_n) + \alpha_n(T^n x_n - Tu_n).$$

That is

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\| \\
&\le (1-\alpha_n)\|x_n - Tu_n\| + \alpha_n \|T^n x_n - Tu_n\| \\
&= (1-\alpha_n)\|x_n - Tu_n\| + \alpha_n \|T^n x_n - T^n u_n\| + \alpha_n \|T^n u_n - Tu_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + (1-\alpha_n)\|u_n - Tu_n\| + \alpha_n k_n \|x_n - u_n\| \\
&\quad + \alpha_n \|T^n u_n - Tu_n\| \\
&\le (1-\alpha_n(1-k_n))\|x_n - u_n\| + (1-\alpha_n)\|u_n - Tu_n\| + \alpha_n \|T^n u_n - Tu_n\| \\
&= (1-\alpha_n(1-k_n))\|x_n - u_n\| + (1-\alpha_n)\|u_{n+1} - u_n\| + \alpha_n \|u_{2n} - u_{n+1}\|.
\end{aligned}$$

Denoting again by

$$\rho_n := \|x_n - u_n\|, \ \lambda_n := \alpha_n(1-k_n) \in (0,1), \ \sigma_n := (1-\alpha_n)\|u_{n+1} - u_n\|, \quad \forall n \in \mathbb{N},$$

we get (4). The assumptions of *Lemma 1* are fulfilled, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Knowing $\lim_{n\to\infty} u_n = x^*$, we get $\lim_{n\to\infty} x_n = x^*$, because

$$\|x_n - x^*\| \le \|x_n - u_n\| + \|u_n - x^*\|, \forall n \ge n_0.$$

Conversely, we suppose that Mann iteration converges to x^*, and we prove that Picard iteration converges to x^*. One can see that

$$\begin{aligned}
\|u_{n+1} - x_{n+1}\| &= \|(1-\alpha_n)(x_n - Tu_n) + \alpha_n(T^n x_n - Tu_n)\| \\
&\le (1-\alpha_n)\|x_n - Tu_n\| + \alpha_n \|T^n x_n - T^n u_n\| + \alpha_n \|T^n u_n - Tu_n\| \\
&\le (1-\alpha_n)\|x_n - Tu_n\| + \alpha_n k_n \|x_n - u_n\| + \alpha_n \|T^n u_n - Tu_n\| \\
&\le \alpha_n k_n \|x_n - u_n\| + (1-\alpha_n)\left[\|x_n - x^*\| + \|x^* - Tu_n\|\right] \\
&\quad + \alpha_n \|T^n u_n - Tu_n\| \\
&\le \alpha_n k_n \|x_n - u_n\| + (1-\alpha_n)\left[\|x_n - x^*\| + \|x^* - Tu_n\|\right] \\
&\quad + \alpha_n \|T^n u_n - Tu_n\| \\
&\le \alpha_n k_n \|x_n - u_n\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)L^n \ \|x^* - u_1\| \\
&\quad + \alpha_n \|T^n u_n - T^n u_1\| \\
&\le \alpha_n k_n \|x_n - u_n\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)L^n \ \|x^* - u_1\| \\
&\quad + \alpha_n k_n \|u_n - u_1\| \\
&\le \alpha_n k_n \|x_n - u_n\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)L^n \ \|x^* - u_1\| \\
&\quad + \alpha_n k_n \left(\|Tu_n\| + \|u_1\|\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n k_n \|x_n - u_n\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)L^n \ \|x^* - u_1\| \\
&\quad +\alpha_n k_n (\|T^n u_1\| + \|u_1\|) \\
&\leq \alpha_n k_n \|x_n - u_n\| + (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)L^n \ \|x^* - u_1\| \\
&\quad +\alpha_n k_n (k_n \|u_1\| + \|u_1\|).
\end{aligned}$$

Denoting by

$$\begin{aligned}
a_n &:= \|x_n - u_n\|, \\
\beta_n &:= (1-\alpha_n)\|x_n - x^*\| + (1-\alpha_n)L^n \ \|x^* - u_1\| + \alpha_n k_n (1+k_n)\|u_1\|, \forall n \in \mathbb{N},
\end{aligned}$$

we get a nonnegative sequence $(a_n)_n$ which verifies

$$a_{n+1} \leq \alpha_n L a_n + \beta_n, \quad \forall n \in \mathbb{N}.$$

We have $L \in (0,1) \Rightarrow \lim_{n\to\infty} L^n = 0$; also, we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$, and consequently $\lim_{n\to\infty} \beta_n = 0$. It is easy to see that

$$\lim_{n\to\infty} a_n = 0, \text{ i.e. } \lim_{n\to\infty} \|x_n - u_n\| = 0.$$

We consider now the proof of the converse. For an $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$, we have

$$\|u_n - x_n\| < \frac{\varepsilon}{2}, \ \|x_n - x^*\| < \frac{\varepsilon}{2}.$$

Finally, we get

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \|u_n - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ i.e. } \lim_{n\to\infty} u_n = x^*.$$

□

Theorem 4. *Let X be a normed space, and B a nonempty convex bounded subset of X. Let $T : B \to B$ be an asymptotic nonexpansive map with $k_n \in (0,1), \forall n \in \mathbb{N}$. Suppose that there exists $x^* \in B$ such that $Tx^* = x^*$. Let $x_1 = v_1 \in B$. The following two assertions are equivalent:*

(i) The Mann iteration $(x_n)_n$ given by (4) strongly converges to x^*,

(ii) The Ishikawa iteration $(v_n)_n$ given by (5) strongly converges to x^*.

Proof. The implication $(ii) \Rightarrow (i)$ is obvious taking $\beta_n = 0, \forall n \in \mathbb{N}$ in (5). We prove the other implication. The following observation will be crucial:

$$\begin{aligned}
&\|x_{n+1} - v_{n+1}\| \\
&= \|(1-\alpha_n)(x_n - v_n) + \alpha_n(T^n x_n - T^n w_n)\| \\
&\leq (1-\alpha_n)\|x_n - v_n\| + \alpha_n \|T^n x_n - T^n w_n\| \\
&\leq (1-\alpha_n)\|x_n - v_n\| + \alpha_n \ k_n \ \|x_n - w_n\| \\
&= (1-\alpha_n)\|x_n - v_n\| + \alpha_n \ k_n \ \|(1-\beta_n)(x_n - v_n) + \beta_n(T^n x_n - v_n)\| \\
&\leq (1-\alpha_n)\|x_n - v_n\| + \alpha_n \ k_n \ (1-\beta_n)\|x_n - v_n\| + \alpha_n \ \beta_n \ k_n \ \|T^n x_n - v_n\| \\
&\leq (1-\alpha_n)\|x_n - v_n\| + \alpha_n \ k_n \ (1-\beta_n)\|x_n - v_n\| + \alpha_n \ \beta_n \ k_n \ (\|T^n x_n\| + \|v_n\|) \\
&\leq [1 - \alpha_n (1 - k_n \ (1-\beta_n))] \|x_n - u_n\| + \alpha_n \ \beta_n \ k_n \ M.
\end{aligned}$$

Taking $\rho_n := \|x_n - v_n\|, \lambda_n := \alpha_n (1 - k_n (1 - \beta_n)) \in (0,1)$, $\sigma_n := \alpha_n \beta_n k_n M$, $\forall n \in \mathbb{N}$, we get relation (7) from *Lemma 1*. Also all assumptions are fulfilled. Thus we get $\lim_{n\to\infty} \rho_n = 0$. We get the conclusion if we regard the following

$$0 \leq \|v_n - x^*\| \leq \|x_n - x^*\| + \|x_n - v_n\| \to 0, (n \to \infty).\square$$

Theorem 3 and *Theorem 4* lead us to the following result:

Corollary 2. *Let X be a normed space, and B a nonempty convex bounded subset of X. Let $T : B \to B$ be an asymptotic nonexpansive map with $k_n \in (0,1)$. Suppose that there exists $x^* \in B$ such that $Tx^* = x^*$. Let $p_1 = x_1 = v_1 \in B$. If the Picard iteration $(u_n)_n$ given by (2) strongly converges to x^*, and $\|u_{n+1} - u_n\| = o(\alpha_n)$, then the Mann sequence $(x_n)_n$ given by (4) strongly converges to x^* and the Ishikawa iteration $(v_n)_n$ given by (5) also strongly converges to x^*. Conversely, if the Mann sequence $(x_n)_n$ given by (4) strongly converges to x^* or the Ishikawa iteration $(v_n)_n$ given by (5) strongly converges to x^*, then the Picard iteration $(u_n)_n$ strongly converges to x^*.*

All our results hold for a set-valued map provided that this map admits appropriate selections.

References

[1] S. S. Chang, Y. J. Cho, B. S. Lee, J. S. Jung, S. M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl. **224**(1998), 149–165.

[2] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44**(1974), 147-150.

[3] L.-S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194**(1995), 114-125.

[4] W. R. Mann, *Mean value in iteration*, Proc. Amer. Math. Soc. **4**(1953), 506-510.

[5] S. A. Mutangadura, C. E. Chidume, *An example on the Mann iteratiuon method for Lipschitz pseudocontraction*, Proc. Amer. Math. Soc. **129**(2001), 2359-2363.

[6] J. A. Park, *Mann-iteration for strictly pseudocontractive maps*, J. Korean Math. Soc. **31**(1994), 333-337.

[7] B. E. Rhoades, Ş. M. Şoltuz, *On the equivalence of Mann and Ishikawa iteration methods*, International Journal of Mathemathics and Mathematical Science, to appear.

[8] Ş. M. Şoltuz, *Some sequences supplied by inequalities and their applications*, Revue d' analyse numerique et de theorie de l'approximation **29**(2000), 207-212.

[9] X. Weng, *Fixed point iteration for local strictly pseudocontractive mapping*, Proc. Amer. Math. Soc. **113**(1991), 727-731.

[10] H. Zhou, J. Yuting, *Approximation of fixed points of strongly pseudocontractive maps without lipschitz assumption*, Proc. Amer. Math. Soc. **125** (1997), 1705-1709.